Trace Ideals over Numerical Semigroup Rings

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Abstract

Numerical semigroups have been studied formally starting in the early 1900s by mathematician Ferdinand Georg Frobenius (1849-1917). Every numerical semigroup can be enriched to have a ring structure called a numerical semigroup ring. Numerical semigroup rings are a more recent object of study and have applications in algebraic geometry as quotients of toric ideals and appear frequently in combinatorial algebra. In this project we study semigroup rings using trace ideals. Trace ideals are a generalization of the trace property and provide a rich theory to detect properties of rings and modules. Our research studies the structure of trace ideals of numerical semigroup rings with few generators. We proved three propositions concerning trace ideals of monomial ideals of numerical semigroup rings.

Numerical Semigroups

0	1	2	3	4	5
6	7	8	9	10	11
12	13	14	15	16	17
18	19	20	21	22	23
24	25	26	27	28	29
30	31	32	33	34	35
36	37	38	39	40	41
42	43	$\boxed{44}$	45	46	47
48	49	50	51	52	53

Figure 1:The numerical semigroup $\langle 6, 9, 20\rangle$.

- A numerical semigroup $(S, +)$ is a submonoid of $(\mathbb{N}, +)$ with finite complement in $\mathbb{N}$.
- The elements in the set $G(S) = \mathbb{N} \setminus S$ are known as the gaps of S.
- The Frobenius number $F(S)$ is the greatest integer not in S.
- The conductor is the least integer $x \in S$ such that $x + n \in S$ for all $n \in \mathbb{N}$. The conductor is equal to $F(S) + 1$.

Numerical Semigroup Rings

Let S be a numerical semigroup and K be any field. The semigroup ring $K[S]$ is the K-subalgebra of the polynomial ring $K[t]$ generated by the monomials t^s where $s \in S$.

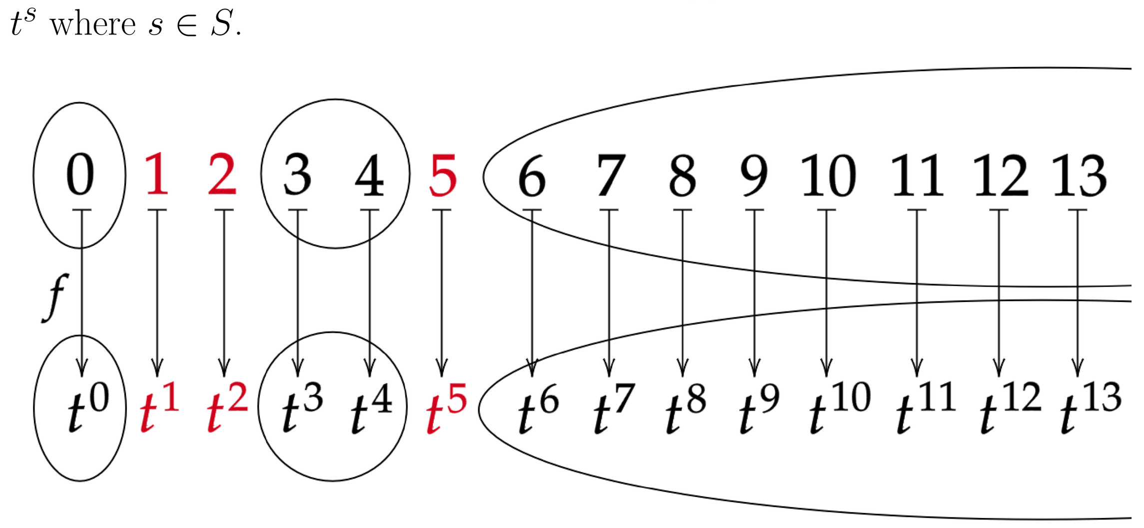

Figure 2:The numerical semigroup $\langle 3, 4\rangle$ mapping into the polynomial ring $K[t]$.

Ideals of Numerical Semigroups

Let $S = \langle 6, 9, 20\rangle$, so that $\mathbb{Q}[S] \simeq \mathbb{Q}[t^6, t^9, t^{20}]$. Consider the ideal $I = (t^6, t^9)$ of $\mathbb{Q}[S]$. What do the elements of I look like? Some examples of elements of I are $t^6, t^{15}, t^{18} + t^9$ and so on. In fact, only exponents of the form $6 + s$ and $9 + s$ with $s \in S$ appear in the ideal I.

Trace Ideal

Let R be a commutative ring with unity and M a module over R. The trace ideal denoted $\tau_M(R)$ is defined as:

$$\tau_M(R) = \sum_{\alpha \in \mathrm{Hom}_R(M,R)} \alpha(M)$$

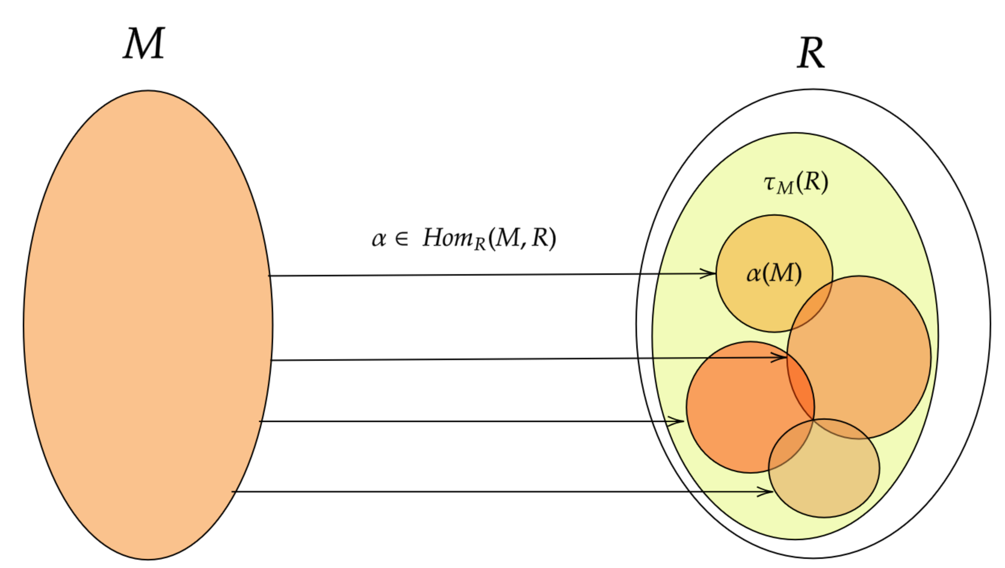

Figure 3:The trace ideal of M over R.

The trace ideal reflects important structural features of its module and conveys large amounts of useful information. For example, they can detect important properties of rings such as the Gorenstein property and also detect free summands. Furthermore, trace ideals are easy to calculate and behave well under many standard operations.

Computing Trace Ideals

Suppose that

$$K[S]^m \xrightarrow{[I]} K[S]^n \xrightarrow{\pi} I \longrightarrow 0$$

is exact so that $[I]$ is the presentation matrix of I. Let $[\varphi]$ be a $1 \times n$ matrix representing a map from R^n to R.

$$\begin{tikzcd} K[S]^m \arrow[r, "{[I]}"] & K[S]^n \arrow[rr, "{[\varphi]}"] \arrow[dr, "\pi"'] & & K[S] \\ & & I \arrow[ur, "\alpha"'] & \end{tikzcd}$$

- Observe that $[\varphi] = \alpha \circ \pi$ if and only if $[\varphi][I] = [0]$ (Vasconcelos [5]).
- Thus, $\mathrm{Hom}(I, K[S])$ can be identified with $\ker([I]^T)$.
- If $[\theta]$ is matrix whose columns generate $\ker([I]^T)$ then $\tau_I(K[S])$ is generated by the entries of $[\theta]$.

This process can be done in Macaulay2 which allows us to automate the computation of trace ideals of semigroup rings.

Example of trace ideals

Consider the numerical semigroup $S = \langle 3, 4\rangle$ and let $I \subseteq K[S]$ be an ideal of $K[S]$. We list the trace ideal $\tau_I(K[S]$ and the structure of $\mathrm{Hom}_{K[S]}(I, K[S])$ below,

I	$\tau_I(K[S])$	$\mathrm{Hom}_{K[S]}(I, K[S])$
(t^7)	(1)	$\{\frac{a}{t^7} + b \mid a, b \in K[S]\}$
(t^3, t^4)	(t^3, t^4)	$\{a \mid a \in K[S]\}$
(t^3, t^8)	(t^3, t^4)	$\{at + b \mid a, b \in K[S]\}$
(t^6, t^8)	(t^6, t^4)	$\left\{\frac{a}{t^2} + b \mid a, b \in K[S]\right\}$

Motivating Questions

Given the following:

- $S = \langle a_1, \ldots, a_n\rangle$ a numerical semigroup.
- K a field.

We ask the questions:

1. *What are the trace ideals of $K[S]$?*
2. *How are the properties of S related to the trace ideals of $K[S]$?*

Conductor Proposition

Let $S = \langle a, b\rangle$ be a numerical semigroup and $I = (t^a, t^d) \subseteq K[[S]]$. If $d > F(S)$ and $d - a \notin S$ then $\tau_I(K[[S]]) = (t^a, t^b)$.

Proof.
For a two-generated numerical semigroup, the Frobenius number has the following form

$$F(S) = ab - a - b.$$

Since $d \geq ab$, we have that

$$d - a \geq ab - a > ab - a - b = F(S)$$

We have shown that $d - a \in S$ and we conclude that I is a principal ideal of $K[S]$ since t^d is a $K[S]$-multiple of t^a.

Maximal Ideal Proposition

Let $S = \langle a_1, \ldots, a_n\rangle$ with $n \geq 2$. The ideal $I = (t^{a_1}, \ldots, t^{a_n}) \subseteq K[[S]]$ is a trace ideal of $K[[S]]$.

Proof Sketch.

- Note: $I \subseteq \tau_I(K[[S]])$.
- I is a maximal ideal of $K[[S]]$ hence $\tau_I(K[[S]]) = I$ or $\tau_I(K[[S]]) = K[[S]]$.
- $K[[S]]$ is local therefore $\tau_I(K[[S]]) = K[[S]] \Rightarrow I$ is a principal ideal.
- However, I is not principal by assumption.
- We conclude that $\tau_I(K[[S]]) = I$.

Corollary to Maximal Ideal Proposition

Let $S = \langle a_1, \ldots, a_n\rangle$ with $n \geq 2$ be a numerical semigroup and $I \subseteq K[[S]]$ be a non-principal ideal of $K[[S]]$. If $(t^{a_1}, \ldots, t^{a_n}) \subseteq \tau_I(K[[S]])$ then $\tau_I(K[[S]]) = (t^{a_1}, \ldots, t^{a_n})$.

Power to Ring Proposition

Let $S = \langle a_1, \ldots, a_n\rangle$ be a numerical semigroup. Denote $I_m = ((t^{a_1})^m, (t^{a_2})^m, \ldots, (t^{a_{n-1}})^m, (t^{a_n})^m)$. There exists some $M \in \mathbb{N}$ such that $m > M$ implies $\tau_{I_m}(K[S]) = K[S]$.

Proof Sketch.

- There exists a large enough $M \in \mathbb{N}$ such that $M \cdot (a_i - a_j) > F(S)$ for all $1 \leq i, j \leq n$ with $j > i$.
- $m > M$ implies that $m \cdot (a_i - a_j) \in S$ for all $1 \leq i, j \leq n$.
- Fix some a_k then $ma_i - ma_k \in S$ for every a_i.
- Consequently, I_m is a principal ideal of $K[S]$ and $\tau_{I_m}(K[S]) = K[S]$.

Methods

We utilized Macaulay2 to compute trace ideals over semigroup rings. Macaulay2 is a mathematics software system for computational algebraic geometry and commutative algebra. It has core algorithms for computing trace ideals including calculating free resolutions of modules and other important linear algebra algorithms. We used two important functions for this project to define semigroup rings and compute their trace ideals. We created the semigroup rings code and modified the trace ideals code in Maitra's paper. The code is shown below.

```
semigroupRing = H -> (S = QQ[t];
                      P := for i in H list t^i;
                      R = QQ[x_(1)..x_(length H)];
                      F = map(S, R, P);
                      return coimage F)

traceIdeal = J -> (C := res J;
                   Am := C.dd_2;
                   trJ = minors(1, syz(transpose Am));
                   return trJ)
```

Future Work

All of our conjectures focus on ideals generated by monomials. We chose to focus on these ideals because the homomorphisms from these ideals to the semigroup ring are often generated by simple fractions. Furthermore, it is possible to relate the possible homomorphisms of monomial ideals to the gaps of the underlying numerical semigroup. The next step in our work is to develop techniques to classify the trace ideals of non-monomial ideals. There is recent development in the theory of trace ideals of semigroup rings by Koboyashi et. al. [7] and we hope to apply this theory to understand the structure of trace ideals of non-monomial ideals.

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